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Higher-order modulation effects on solitary wave envelopes in deep water

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The envelope equation of Dysthe (1979), which provides an extension of the nonlinear Schrödinger equation (NLS) to fourth order in wave steepness, is used to discuss higher-order modulation effects on the long-time evolution of solitary wave envelopes in deep water. The Dysthe equation admits solitary-wave solutions, similar to those of the NLS. Using perturbation methods, it is shown that an initial disturbance in the form of a solitary wave group of the NLS evolves to a solitary wave of the Dysthe equation having lower peak amplitude and moving with higher speed than the original wave; the increase in wave speed is caused by a downshift in wavefrequency. Asymptotic expressions are derived for this amplitude decrease and frequency downshift, which are consistent with numerical and experimental results.

1. Introduction

The propagation of the envelope of a weakly nonlinear wavepacket in deep water is governed asymptotically by the nonlinear Schrödinger equation (NLS), which includes the leading-order nonlinear and dispersive effects (see, for example, Yuen & Lake 1982). On theoretical grounds, the NLS is expected to provide an accurate description of the evolution of a wavepacket of small steepness ϵ only for a limited time, at most $O(\epsilon^{-2})$ wave periods. This restriction on the validity of the NLS has been confirmed experimentally: Feir (1967) reports that an initially symmetric wavepacket of uniform frequency and moderate steepness eventually loses its symmetry as it propagates away from the wavemaker, and splits into two distinct groups of different frequencies. More recently, similar phenomena were observed by Su (1982), in systematic experiments with initially symmetric wavepackets of uniform frequency and various durations. Depending on the initial wave steepness and wavepacket duration, several (up to seven) distinct wave groups, which resembled solitary wave groups of the NLS, developed far from the wavemaker; furthermore, owing to a relative downshift in its carrier frequency and hence an increase in group velocity, the leading group clearly separated from the rest of the disturbance. On the other hand, the NLS predicts that an initially symmetric wavepacket of uniform frequency will always remain symmetric and, in general, will form a bound state of solitons which, rather than separate, exhibit recurrence phenomena. Thus the need for a more accurate theoretical treatment is evident.

A first attempt to predict the observations of Feir (1967) theoretically was made by Roskes (1977), using the NLS, modified with some additional terms representing higher-order modulation effects. It is noteworthy that, although, as it turns out, Roskes (1977) neglected the interaction of the wave envelope with the induced mean flow, his numerical calculations were capable of reproducing, at least qualitatively, the group splitting observed by Feir (1967). Later, Dysthe (1979) carried out a formal derivation of an envelope equation which extends the range of validity of the NLS to longer time, comparable with $O(\epsilon^{-3})$ wave periods; in addition to the higher-order terms proposed by Roskes (1977), Dysthe's equation also includes the effect of the wave-induced mean flow. Lo & Mei (1985) conducted a detailed numerical study of the long-time evolution of short wavepackets using the full Dysthe equation, and found good quantitative agreement with the experiments of Su (1982). In particular, they confirmed that wave-group separation is caused by a frequency downshift in the leading group, and that the resulting wave groups have envelopes similar to the familiar 'sech' solitary-wave profiles of the NLS. Furthermore, based on their numerical solutions of the Dysthe equation, Lo & Mei (1985) (see also Lo 1986) pointed out that wave envelopes, which initially are in the form of a single stationary soliton of the NLS (in a frame moving with the group velocity), propagate with a finite speed which is an increasing function of the initial maximum wave amplitude.

By assumption, the higher-order modulation terms in the Dysthe equation are relatively small and, therefore, their effect is expected to become important only after a long time. This suggests that one should be able to make use of perturbation methods, in order to describe the experimentally observed deviations from the predictions of the NLS. Adopting this point of view, which has also proven useful in dealing with perturbed evolution equations in other instances (see, for example, Kodama & Ablowitz 1981), the present paper is concerned with an asymptotic study of the long-time evolution of deep-water wavepackets, based on the Dysthe equation. In the same spirit, Janssen (1983) has already developed a nonlinear stability theory for periodic modulations of a uniform wavetrain near the threshold for sideband instability. Here, however, attention is focused on localized wavepackets. More specifically, a perturbation expansion for small wave steepness indicates that the Dysthe equation admits solitary-wave solutions, similar to those of the NLS; a numerical continuation procedure shows that this is also the case for moderate values of the wave steepness. In addition, motivated by the experimental and numerical results cited above, an initial-value problem is solved asymptotically, using a stationary soliton of the NLS as initial condition. This initial disturbance eventually transforms to a solitary wave of the Dysthe equation, having lower peak amplitude and moving with higher speed than that of the original wave, in agreement with the numerical results of Lo & Mei (1985); the increase in wave speed is caused by a downshift in wave frequency, proportional to the square of the wave amplitude, and turns out to be independent of the wave-induced mean flow, to leading order in wave steepness. These asymptotic results are consistent with the experiments of Feir (1967) and Su (1982) and support the explanation of wave-group separation, proposed by Roskes (1977) and Lo & Mei (1985).

2. Solitary wave envelopes

Consider a two-dimensional modulated wavepacket of carrier wavenumber k_0, carrier frequency ω_0, and small steepness ϵ $(0 < \epsilon \ll 1)$, propagating on deep water $(-\infty < x < \infty, -\infty < y < 0)$. In dimensionless variables, using $1/k_0$ as a lengthscale and $1/\omega_0$ as a timescale, the free-surface elevation takes the form

$$y = \tfrac{1}{2}\epsilon A(X,T)\exp[\mathrm{i}(x-t)] + * + O(\epsilon^2),$$

where A is the complex wave envelope, which depends on the 'slow' variables $X = \epsilon x$, $T = \epsilon t$, and $*$ denotes the complex conjugate.

From the perturbation analysis of Dysthe (1979), correct to $O(\epsilon^4)$, it follows that A is governed by the evolution equation

$$A_T + \tfrac{1}{2}A_X + \mathrm{i}\epsilon(\tfrac{1}{8}A_{XX} + \tfrac{1}{2}A^2A^*) + \epsilon^2(-\tfrac{1}{16}A_{XXX} + \tfrac{3}{2}AA^*A_X - \tfrac{1}{4}A^2A^*_X + \tfrac{1}{2}\mathrm{i}A\mathscr{H}\{AA^*\}_X) = 0, \tag{1}$$

where $\mathscr{H}$ stands for the Hilbert transform

$$\mathscr{H}\{A\} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{A(\tau)}{\tau - X}\,\mathrm{d}\tau,$$

the integral being interpreted as a principal value. According to (1), the wave envelope propagates with the group velocity (equal to $\frac{1}{2}$ in dimensionless variables) and is slowly modulated by weak dispersive and nonlinear effects. In particular, the $O(\epsilon)$ terms in (1) represent the leading-order dispersive and nonlinear effects included in the NLS, while the $O(\epsilon^2)$ terms are the higher-order corrections given by Dysthe (1979); the effect of the wave-induced mean flow is described by the last term in (1). It is convenient to adopt a frame of reference moving with the group velocity:

$$\xi = 2X - T, \quad \eta = \epsilon X, \tag{2}$$

so that the envelope equation (1), correct to the same order of approximation, becomes

$$A_\eta + \mathrm{i}A_{\xi\xi} + \mathrm{i}A^2A^* + 8\epsilon AA^*A_\xi + 2\mathrm{i}\epsilon A\mathscr{H}\{AA^*\}_\xi = 0. \tag{3}$$

As noted by Lo & Mei (1985), the transformation (2) simplifies Dysthe's equation substantially; also, this form of the evolution equation is more suited for discussing the spatial evolution of a wavepacket, generated by a wavemaker at a fixed X-location, as in the experiments of Su (1982).

As is well known, the NLS, which is obtained from (3) by neglecting all $O(\epsilon)$ terms, admits solitary-wave solutions with a 'sech' profile. It is of interest, therefore, to examine the possibility that the higher-order equation (3) has similar solutions; to this end, following Roskes (1977), we write

$$A = r(\Theta;\epsilon)\exp[\mathrm{i}(\psi + \epsilon f(\Theta;\epsilon))], \tag{4}$$

where

$$\Theta = \kappa\xi - \lambda\eta, \quad \psi = \mu\xi - \sigma\eta. \tag{5}$$

Upon direct substitution of (4) into (3), it is found that

$$\lambda = -2\kappa\mu, \tag{6}$$

$$f_\Theta = \frac{2}{\kappa}r^2, \tag{7}$$

and

$$r_{\Theta\Theta} - \alpha^2 r + \frac{1}{\kappa^2}r^3 + \frac{\epsilon}{\kappa^2}[8\mu r^3 + 2\kappa r\mathscr{H}\{r^2\}_\Theta] + \frac{12\epsilon^2}{\kappa^2}r^5 = 0, \tag{8}$$

where

$$\alpha^2 = \frac{\mu^2 + \sigma}{\kappa^2}.$$

From (6) it is clear that higher-order effects do not modify the propagation speed of possible solitary-wave solutions; they merely give rise to $O(\epsilon^2)$ changes in wave frequency and wavenumber, which can be readily calculated from (7) if the envelope profile $r(\Theta)$ is known.

To establish the existence of solitary waves, it is necessary to find special solutions of the nonlinear integral–differential equation (8), which decay as $\Theta \rightarrow \pm\infty$; as this task appears to be rather difficult for arbitrary values of ϵ, we resort to a perturbation expansion for small ϵ:

$$r(\Theta;\epsilon) = r_0(\Theta) + \epsilon r_1(\Theta) + \epsilon^2 r_2(\Theta) + \dots, \tag{9}$$

where

$$r_0(\Theta) = a \operatorname{sech} \alpha\Theta, \tag{10}$$

with $a = \sqrt{2\alpha\kappa}$, is the familiar solitary-wave profile of the NLS. Proceeding to $O(\epsilon)$, it is found that r_1 satisfies the inhomogeneous problem:

$$\mathscr{L} r_1 = \mathscr{R}_1, \tag{11}$$

where

$$\mathscr{R}_1 = 16a(2\kappa Sh - \mu S^3), \tag{12a}$$

$$h(\Theta) = \frac{\alpha}{\pi^3} \int_0^\infty k^2 \frac{\cos(2k\alpha\Theta/\pi)}{\sinh k} \mathrm{d}k, \tag{12b}$$

$\mathscr{L}$ is the linear operator

$$\mathscr{L} \equiv \frac{1}{\alpha^2}\frac{\mathrm{d}^2}{\mathrm{d}\Theta^2} + 6\operatorname{sech}^2\alpha\Theta - 1, \tag{13}$$

and S is a shorthand notation for $\operatorname{sech}\alpha\Theta$. The homogeneous part of (11) has two linearly independent solutions

$$f_1 = SR, \quad f_2 = \alpha\Theta SR + \frac{1}{3S} - S, \tag{14}$$

were $R \equiv \tanh\alpha\Theta$; since f_2 is unbounded as $\Theta \rightarrow \pm\infty$, the right-hand side of (11) has to be orthogonal to f_1, in order for the solution of (11) to decay as $\Theta \rightarrow \pm\infty$:

$$\int_{-\infty}^{\infty} f_1 \mathscr{R}_1 \,\mathrm{d}\Theta = 0. \tag{15}$$

This solvability condition, which is trivially met here because f_1 is an odd function of Θ while $\mathscr{R}_1$ is even, ensures that $r_1 \rightarrow 0$ as $\Theta \rightarrow \pm\infty$. Also, by rotating the path of integration to the positive imaginary k-axis and using Watson's lemma, the integral in (12b) can be evaluated asymptotically:

$$h(\Theta) \sim -\frac{1}{4\pi\alpha\Theta^2} \quad (|\Theta| \rightarrow \infty);$$

therefore, since $\mathscr{R}_1$ vanishes faster than $\mathrm{e}^{-\alpha|\Theta|}$ as $|\Theta| \rightarrow \infty$, $r_1 = O(\epsilon^{-\alpha|\Theta|})$ and the expansion (9) remains well-ordered as $|\Theta| \rightarrow \infty$.

In the next order of approximation $O(\epsilon^2)$, r_2 satisfies

$$\mathscr{L} r_2 = \mathscr{R}_2, \tag{16}$$

where

$$\mathscr{R}_2 = -\frac{1}{\alpha^2\kappa^2}[3r_0 r_1^2 + 24\mu r_1 r_0^2 + 4\kappa r_0 \mathscr{H}\{r_0 r_1\}_\Theta + 2\kappa r_1 \mathscr{H}\{r_0^2\}_\Theta + 12 r_0^5].$$

Again, as it is an even function of Θ, $\mathscr{R}_2$ is orthogonal to f_1 so that $r_2 \rightarrow 0$ as $|\Theta| \rightarrow \infty$; also, $\mathscr{R}_2 = o(\mathrm{e}^{-\alpha|\Theta|})$ and no non-uniformities arise at infinity.

Formally at least, the above perturbation procedure can be continued to higher orders in ϵ, and it indicates that the envelope equation (3) admits symmetric solitary-wave solutions with exponential tails ($r = O(\mathrm{e}^{-\alpha|\Theta|})$, $|\Theta| \rightarrow \infty$). This claim is supported by numerical calculations of such solutions of the nonlinear integral–differential

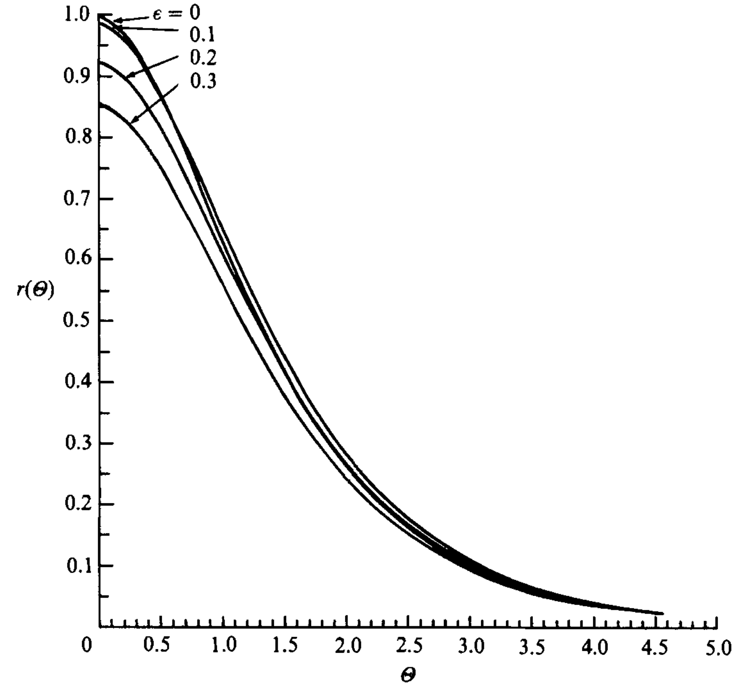

FIGURE 1. Stationary solitary-wave profiles of the Dysthe equation, $r(\Theta)$, for various values of the wave steepness ϵ.

equation (8) for finite values of ϵ. The numerical method of solution is based on a pseudospectral approximation. We look for waves which are symmetric about $\Theta = 0$; for this purpose, the semi-infinite domain $0 \leqslant \Theta \leqslant \infty$ is truncated to a suitably large but finite domain $0 \leqslant \Theta \leqslant \Theta_\infty$, say, which forms the computational period and is discretized by $N+1$ evenly spaced points. The values of r at these points are interpolated by a trigonometric polynomial of degree N and then $r_{\Theta\Theta}$ is approximated by differentiating this polynomial; similarly, the Hilbert transform in (8) is computed using discrete Fourier transforms. Thus, a set of $N+1$ nonlinear algebraic equations is obtained for the values of r at the grid points. For given μ, α, κ and ϵ, the corresponding solitary-wave profile is computed by solving this set of equations numerically, using Newton's method, combined with continuation in ϵ, starting from $\epsilon = 0$ with the known solitary-wave solution (10) of the NLS.

Detailed computations were carried out for stationary envelopes ($\mu = 0$) with $\alpha = 1, \kappa = 1/\sqrt{2}$; for these parameter values, the solitary wave of the NLS ($\epsilon = 0$) has peak amplitude equal to 1. Figure 1 shows the effect of increasing ϵ on the wave profile. For ϵ less than 0.1, the numerical results are in very good agreement with the small-steepness expansion (9) (correct to $O(\epsilon^2)$), and the corresponding waves are practically indistinguishable from the solitary wave of the NLS. For larger values of ϵ, higher-order effects are more evident and cause the peak wave amplitude to decrease slightly. In these computations, the specific values $N = 128$, $\Theta_\infty = 15$ were used, after having verified that increasing this resolution had no appreciable effect on the numerical results. As an additional check, Dysthe's equation (3) was solved numerically by the split-step Fourier method of Lo & Mei (1985) for $\eta > O(\epsilon^{-1})$, using the computed stationary wave envelopes as initial conditions; it was confirmed that these waves are indeed solutions of permanent form, with error less than 5 %.

3. Initial-value problem

Motivated by earlier experimental and numerical results indicated in §1, here we examine the role of higher-order modulation effects in the long-time evolution of a stationary solitary wave of the NLS, using perturbation methods for small wave steepness ϵ. In particular, we solve Dysthe's equation (3) subject to the initial condition

$$A = a_0 \operatorname{sech} \alpha_0 \theta \quad (\eta = 0), \tag{17}$$

where $\theta = \kappa\xi$, $a_0 = \surd 2\alpha_0 \kappa$. It proves convenient for the following analysis to write

$$A(\xi, \eta; \epsilon) = U(\xi, \eta; \epsilon) \exp(-\mathrm{i}\tfrac{1}{2}a_0^2 \eta), \tag{18}$$

and work with U rather than A. For small ϵ, it is natural to expand U in powers of ϵ:

$$U = U_0(\theta) + \epsilon U_1(\theta, \eta) + \epsilon^2 U_2(\theta, \eta) + \dots, \tag{19}$$

where, in view of (17), (18), U_0 is the profile of a stationary solitary wave of the NLS, $U_0 = a_0 \operatorname{sech} \alpha_0 \theta$; as is customary in perturbation theory, expansion (19) is proposed here keeping in mind that it will later be revised appropriately, in case non-uniformities arise owing to the appearance of secular terms.

Upon substitution of (19) into (3), taking into account (17), it is found that, to $O(\epsilon)$, $U_1 = u_1 + \mathrm{i}v_1$ satisfies a linear inhomogeneous equation, subject to the quiescent initial condition $U_1 = 0$ at $\eta = 0$. Taking Laplace transforms in η,

$$U_1 = \frac{1}{2\pi \mathrm{i}} \int \hat{U}_1 \, \mathrm{e}^{s\eta} \, \mathrm{d}s,$$

this initial-value problem leads to a pair of coupled real equations for $\hat{u}_1, \hat{v}_1$:

$$s\hat{v}_1 + \tfrac{1}{2}a_0^2 \mathscr{L}\hat{u}_1 = -\frac{2\kappa}{s} U_0 \mathscr{H}\{U_0^2\}_\theta, \tag{20a}$$

$$s\hat{u}_1 - \tfrac{1}{2}a_0^2 \mathscr{M}\hat{v}_1 = -\frac{8\kappa}{s} U_0^2 U_{0\theta}, \tag{20b}$$

where $\mathscr{L}$ is the linear operator defined earlier in (13) (with α replaced by α_0 and Θ replaced by θ), and $\mathscr{M}$ stands for the operator

$$\mathscr{M} \equiv \frac{1}{\alpha_0^2} \frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + 2 \operatorname{sech}^2 \alpha_0 \theta - 1.$$

Thus, solving (20) for $\hat{u}_1$, $\hat{v}_1$ and inverting these Laplace transforms yields a formal solution for the $O(\epsilon)$ correction U_1. However, before proceeding to higher order, it is necessary to check whether the expansion (19) remains uniformly valid; for this purpose, the asymptotics of U_1 for large η are needed.

As usual, the asymptotic behaviour of U_1 for large η is deduced from the singularities of $\hat{U}_1$ in the complex s-plane. The form of the right-hand side of (20) suggests that $s = 0$ is a possible singularity, and, for this reason, we examine the local behaviour of $\hat{u}_1$ and $\hat{v}_1$ near $s = 0$. Assuming first that

$$\hat{u}_1 = \frac{1}{s} p_{-1}(\theta) + p_0(\theta) + o(1), \tag{21a}$$

$$\hat{v}_1 = \frac{1}{s} q_{-1}(\theta) + q_0(\theta) + o(1), \tag{21b}$$

and substituting into (20), it is found that to $O(1/s)$

$$\mathscr{L}p_{-1} = 32\kappa a_0 S_0 h_0(\theta), \tag{22a}$$

$$\mathscr{M}q_{-1} = -16\kappa\alpha_0 a_0 R_0 S_0^3; \tag{22b}$$

here $h_0(\theta)$ stands for $h(\Theta)$ defined in (12*b*) (with α replaced by α_0 and Θ replaced by θ), and the shorthand notation $S_0 \equiv \operatorname{sech}\alpha_0\theta$, $R_0 \equiv \tanh\alpha_0\theta$ is used again. From the previous discussion (see §2), it follows that (22*a*) has a well-behaved solution, $g(\theta)$, say, which decays as $|\theta| \to \infty$, since the right-hand side of (22*a*) satisfies the solvability condition (15). Similarly, (22*b*) also has a well-behaved solution:

$$q_{-1} = \frac{2a_0^3}{\alpha_0\kappa} S_0 R_0. \tag{23}$$

Proceeding to $O(1)$, one has

$$\mathscr{L}p_0 = -\frac{2}{a_0^2} q_{-1}, \tag{24a}$$

$$\mathscr{M}q_0 = \frac{2}{a_0^2} p_{-1}. \tag{24b}$$

Now, however, it is clear that (24*a*) does not have an acceptable solution: in view of (23), the right-hand side is proportional to $S_0 R_0$ and the solvability condition (15) cannot be met. Also, it can be shown that the right-hand side of (24*b*) is not orthogonal to the corresponding well-behaved homogeneous solution, S_0, and, thus, no acceptable q_0 can be found either. These difficulties suggest that the assumptions made in (21) are not valid; it turns out that (21) need to be modified as follows:

$$\hat{u}_1 = \frac{C_1}{s^2} S_0 R_0 + \frac{1}{s} p_{-1}(\theta) + p_0(\theta) + o(1), \tag{25a}$$

$$\hat{v}_1 = \frac{C_2}{s^2} S_0 + \frac{1}{s} q_{-1}(\theta) + q_0(\theta) + o(1), \tag{25b}$$

where C_1, C_2 are as yet undetermined constants. Taking into account these changes, p_{-1}, q_{-1} now are given by

$$p_{-1} = \frac{C_2}{a_0^2}(\alpha_0\theta S_0 R_0 - S_0) + g(\theta), \tag{26a}$$

$$q_{-1} = -\frac{C_1}{a_0^2}\alpha_0\theta S_0 + \frac{2a_0^3}{\alpha_0\kappa} S_0 R_0, \tag{26b}$$

and the solvability condition for (24*a*),

$$\int_{-\infty}^{\infty} q_{-1} S_0 R_0 \,\mathrm{d}\theta = 0,$$

specifies the value of C_1:

$$C_1 = \tfrac{8}{3}\alpha_0\kappa a_0^3. \tag{27}$$

Similarly, the solvability condition for (24*b*),

$$\int_{-\infty}^{\infty} p_{-1} S_0 \,\mathrm{d}\theta = 0,$$

which can be reduced to the simpler form

$$C_2 + 32\kappa\alpha_0 a_0^3 \int_0^{\infty} h_0 S_0(\alpha_0\theta S_0 R_0 - S_0)\,\mathrm{d}\theta = 0,$$

determines C_2:

$$C_2 = \frac{32I}{\sqrt{2\pi^3}}\, a_0^4, \tag{28a}$$

where

$$I = \int_0^\infty \frac{k^3}{\sinh^2 k}\,\mathrm{d}k = 1.8031. \tag{28b}$$

According to (25), $\hat{u}_1$, $\hat{v}_1$ have a double pole at $s = 0$. Inverting these Laplace transforms, the asymptotic behaviour of $U = u + \mathrm{i}v$, correct to $O(\epsilon)$, for η large is

$$u \sim a_0 S_0 + \epsilon\left[C_1 \eta S_0 R_0 + \frac{C_2}{a_0^2}(\alpha_0 \theta S_0 R_0 - S_0) + g(\theta)\right] + \ldots, \tag{29a}$$

$$v \sim \epsilon\left(C_2 \eta S_0 - \frac{C_1}{a_0^2}\alpha_0 \theta S_0 + \frac{2a_0^3}{\alpha_0 \kappa} S_0 R_0\right) + \ldots. \tag{29b}$$

Therefore, u_1, v_1 exhibit secular behaviour as $\eta \to \infty$ and the expansion (19) becomes non-uniform for $\epsilon\eta = O(1)$. From the viewpoint of perturbation theory, (29) may be considered as the outer limit of the inner expansion (19), which has to be matched to an appropriate outer expansion, in order to obtain a uniformly valid description for $\epsilon\eta = O(1)$. Here, it turns out that this outer representation is furnished by a particular solitary-wave solution of the Dysthe equation, already discussed in §2.

Indeed, returning to (4), (9), (10), consider an almost stationary ($\mu \ll 1$) solitary wave with slightly different amplitude, a, than that of the original wave:

$$\mu = \beta\epsilon, \quad a = a_0 + \gamma\epsilon \quad (\beta, \gamma = O(1)), \tag{30}$$

so that, from (6)–(8),

$$\lambda = -2\beta\kappa\epsilon, \quad \sigma = \tfrac{1}{2}a_0^2 + a_0\gamma\epsilon + O(\epsilon^2), \quad \alpha = \alpha_0 + \frac{\gamma\alpha_0}{a_0}\epsilon + O(\epsilon^2), \quad \Theta = \theta + 2\beta\kappa\eta\epsilon. \tag{31}$$

Then, a Taylor-series expansion in powers of ϵ yields

$$r_0 = a_0 S_0 - \epsilon a_0\left(2\beta\alpha_0\kappa\eta + \frac{\gamma\alpha_0}{a_0}\theta\right) S_0 R_0 + \epsilon\gamma S_0 + O(\epsilon^2), \tag{32a}$$

$$r_1 = g(\theta) + O(\epsilon), \quad f = \frac{2a_0^2}{\alpha_0\kappa} R_0 + O(\epsilon), \tag{32b}$$

$$\exp\left[\mathrm{i}(\psi + \epsilon f)\right] = \exp\left(-\mathrm{i}\tfrac{1}{2}a_0^2\eta\right)\left[1 + \mathrm{i}\epsilon\left(\frac{\beta}{\kappa}\theta - a_0\gamma\eta + \frac{2a_0^2}{\alpha_0\kappa}R_0\right) + O(\epsilon^2)\right]. \tag{32c}$$

Thus, combining (4), (9), (32), the following expression, correct to $O(\epsilon)$, for the solitary wave is obtained:

$$A = Q\exp\left(-\mathrm{i}\tfrac{1}{2}a_0^2\eta\right), \tag{33a}$$

where

$$Q = a_0 S_0 + \epsilon\left[\gamma S_0 - a_0\left(2\beta\alpha_0\kappa\eta + \frac{\gamma\alpha_0}{a_0}\theta\right) S_0 R_0 + g(\theta)\right] + \mathrm{i}\epsilon\left(\frac{a_0\beta}{\kappa}\theta S_0 - a_0^2\gamma\eta S_0 + \frac{2a_0^3}{\alpha_0\kappa} S_0 R_0\right) + \ldots. \tag{33b}$$

Comparing (18), (29) with (33), taking into account (27), (28), matching to $O(\epsilon)$ is achieved if

$$\beta = -\tfrac{4}{3}a_0^2, \quad \gamma = -\frac{32I}{\sqrt{2\pi^3}}\,a_0^2,$$

and from (30), (31) it follows that

$$\mu = -\tfrac{4}{3}a_0^2\,\epsilon, \quad \lambda = \tfrac{8}{3}\kappa a_0^2\,\epsilon, \quad a = a_0\left(1-\frac{32I}{\sqrt{2\pi^3}}\,a_0\,\epsilon\right). \tag{34}$$

Recalling the definitions of ξ, η in (2), the above values of μ and λ can be interpreted, respectively, as a frequency downshift $\Delta\omega$ and a velocity increase Δc in the original wave, which (in the (x,t)-coordinates) are proportional to the square of the wave steepness:

$$\Delta\omega = -\tfrac{4}{3}a_0^2\,\epsilon^2, \quad \Delta c = \tfrac{2}{3}a_0^2\,\epsilon^2. \tag{35}$$

Therefore, according to the perturbation theory, correct to $O(\epsilon)$, an initial disturbance in the form of a stationary solitary wave of the NLS evolves to a slowly moving solitary wave of the Dysthe equation, having slightly lower peak amplitude and carrier frequency than those of the original wave. In principle, the asymptotic expressions (34), (35) could be refined by carrying the perturbation analysis to higher order in ϵ. However, it should be kept in mind that Dysthe's equation (3) is correct to $O(\epsilon)$ only, and, in order to be consistent with the full water-wave theory, it would be necessary to include higher-order terms in (3) before continuing the perturbation analysis.

The asymptotic expressions (34), (35) are in good agreement with results obtained from numerical solutions of the Dysthe equation (3), subject to the initial condition (17), for small wave steepness ϵ. The numerical calculations were carried out using the split-step Fourier method of Lo & Mei (1985), with a computational period in ξ consisting of 256 grid points and step sizes $\Delta\xi = 0.165$, $\Delta\eta = 5\times10^{-3}$. In the numerical solution, it was most convenient to monitor the evolution of the disturbance by recording the peak amplitude and the speed of the wave envelope in the (ξ,η)-coordinate system which, in accordance with the computations of Lo & Mei (1985), was observed to be an increasing function of ϵ and the initial amplitude a_0. On the other hand, from (5), (34), it follows that this same speed is given asymptotically by

$$\frac{\lambda}{\kappa} = \tfrac{8}{3}a_0^2\,\epsilon. \tag{36}$$

Figure 2 shows a comparison for various values of ϵ of the asymptotic result (36) with estimates of the wave-group speed from numerical solutions of (3), using $a_0 = 1$ in (17). As expected, the agreement becomes better as ϵ is decreased, but even for moderately small ϵ the asymptotic result (36) is reasonable. Similar agreement with numerical results is found for the envelope peak amplitude, which follows the asymptotic formula in (34) quite closely; of course, this is also due to the fact that, as is evident from figure 1, the contribution of $r_1(\Theta)$ to the envelope profile is very small for $\epsilon \leqslant 0.1$ and it is masked by the $O(\epsilon)$ modification of the peak amplitude, given in (34). We also remark that the numerical solutions indicate the presence of small-amplitude dispersive waves behind and ahead of the slowly moving wave envelope (see also Lo 1986); as suggested by the perturbation theory, these waves are most likely transients (represented by the neglected terms in (29)) which, however, eventually reach the ends of the computational domain and can cause appreciable numerical error for moderate values of ϵ. Furthermore, it is interesting to note that

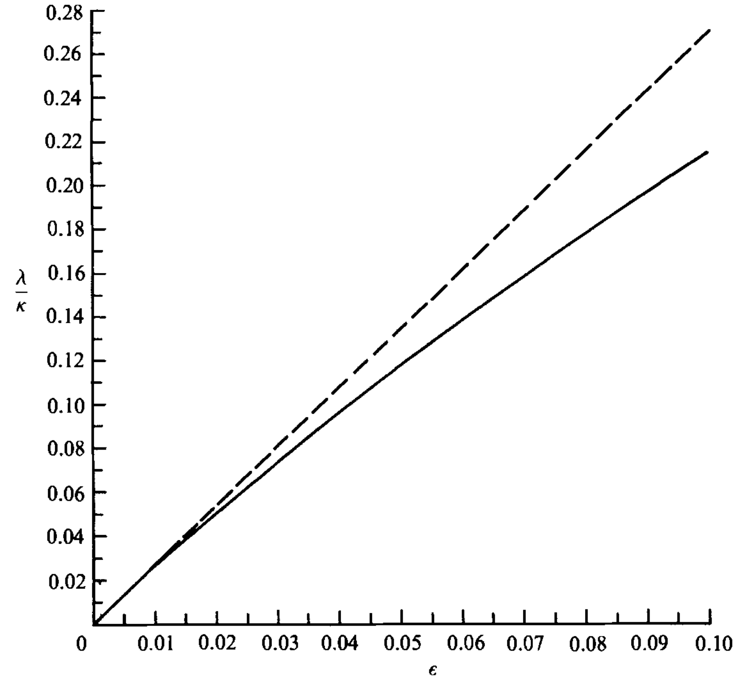

FIGURE 2. Comparison between asymptotic and numerical results for the envelope speed (in the (ξ, η)-coordinate system), λ/κ, as a function of the wave steepness ϵ. ——, numerical; – – –, asymptotic.

the speed change in (35) is independent of the wave-induced mean flow, at least to leading order in ϵ; this suggests an explanation for the wave-group splitting predicted by the numerical calculations of Roskes (1977), even though the mean-flow term of the Dysthe equation was neglected.

The asymptotic results derived in this paper support the explanation, proposed by Roskes (1977) and Lo & Mei (1985), for the wave-group separation observed in the experiments of Feir (1967) and Su (1982). Although the simplest possible initial condition consisting of a single soliton was studied in detail, it is anticipated that, for more general initial conditions so that more than one solitary wave groups are present, higher-order effects will give rise to frequency shifts and speed changes depending on the amplitude of each group, thus causing group splitting. It appears that the asymptotic methods developed here can be extended to study the evolution of more general initial disturbances and thereby obtain explicit expressions for these frequency shifts and speed changes.

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